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A theory of turbulent transport

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A calculation is made of the turbulent transport terms (third moments) that occur in the Reynolds stress equation for buoyant and/or sheared fluids. This calculation is based on neglect of a two-time fourth-order cumulant – a weaker approximation than neglect of the usual single-time fourth-order cumulant. The previously used eddy-damping assumption for single point moments is avoided. This assumption is then examined critically. Comparison is afterward made between the turbulent transport terms derived here and those derived previously by the eddy-damping method, and between the respective derivations. Also the dissipation of third moments is calculated. The calculation is formally limited to mean quantities which vary but slowly in space and time, and to small anisotropy.

1. Introduction

As is well known, second-order modelling (also called single-point modelling) begins with an equation for second-moments in terms of third moments – the Reynolds stress equation. Included among the third moments are turbulent transport terms (diffusion of kinetic energy fluctuations and diffusion of thermal energy fluctuations) and the pressure–strain term. The task of theory is to express these third moments in terms of second-moments – the closure problem. Our article concerns the transport terms. One method for calculating transport terms is based on the eddy-damped quasi-normal (EDQN) approximation as applied to single-point moments (Hanjalić & Launder 1972; Lumley, Zeman & Seiss 1978; Zeman & Lumley 1976; André *et al.* 1978), which differs from the EDQNM as applied to two-point moments (Orszag 1970). This single-point method has had significant success but, more recently, its basic assumptions have been called into question for inhomogeneous flows (Deardorff 1978; Wyngaard 1979; Zeman 1981); e.g. Wyngaard (1979) found a significant discrepancy for eddy damping. The purpose of our paper is to present an alternative method for calculating transport terms and, at the same time, in the course of this calculation, to help assess the validity of the eddy-damping assumption for single-point moments. Specifically, we calculate the third moments in a direct, somewhat more rigorous fashion that avoids that assumption.

The method we use is based on the neglect of a two-time fourth-order cumulant (Chandrasekhar 1955; Kraichnan 1957) an approximation that differs from the quasi-normal neglect of single-time fourth-order cumulants (e.g. Hanjalić & Launder 1972; Proudman & Ried 1954). Discussion of this approximation is given in §2. A similar method was used to calculate the pressure–strain term (Weinstock 1981, 1982, 1986; Weinstock & Burk 1985). Experience with sophisticated turbulence theory is neither required nor expected. Very briefly, this method is based on a

straightforward, formal integration of the Navier–Stokes equation that permits one to express single-time third moments in terms of two-time fourth moments. The two-time fourth moments are then evaluated by neglect of two-time fourth-order cumulants. This procedure will be seen to avoid the basic assumptions inherent in the eddy-damping method. We also hope that this treatment may be accurate as an application of two-point closures since we need mainly the energetic part of spectra at short times where two-point closures are least controversial.

To help clarify the theory and display its own underlying approximations, the following simplifying assumptions are made: (*a*) all average quantities are assumed to vary slowly in space and time compared to L_0 and τ_L, respectively, where L_0 is the characteristic lengthscale of the energy-containing part of the spectrum and τ_L is a Lagrangian timescale; (*b*) small anisotropy; (*c*) average quantities may vary in only the vertical direction (horizontal stratification); (*d*) a unidirectional mean flow of the form $U = [U_0(x_3), 0, 0]$ with Cartesian coordinates $[x_1, x_2, x_3]$.

The paper is organized as follows: the turbulence transport terms (third-moments) are derived in §2, the eddy-damping assumption is discussed in §3, the dissipation of third moments is derived in §4, and a fourth-order cumulant is modelled in §5. Section 6 contains a comparison between our derivation of transport terms and the EDQN derivation, and a summary is given in §7. Details of most proofs are given in Appendices D, E, and F, copies of which can be obtained from the *Journal of Fluid Mechanics* editorial office or the author.

2. Calculation of fluxes

The transport terms that appear in the Reynolds stress equation and in moments of the temperature equation are

$$\left.\begin{aligned} \boldsymbol{\nabla}\cdot\langle \boldsymbol{uuu}\rangle &\equiv \frac{\partial}{\partial x_3}\langle u_3 \boldsymbol{uu}\rangle, \qquad \boldsymbol{\nabla}\cdot\langle \boldsymbol{uu}\theta\rangle \equiv \frac{\partial}{\partial x_3}\langle u_3 \boldsymbol{u}\theta\rangle, \\ \boldsymbol{\nabla}\cdot\langle \boldsymbol{u}\theta^2\rangle &\equiv \frac{\partial}{\partial x_3}\langle u_3 \theta^2\rangle, \qquad \boldsymbol{\nabla}\cdot\langle \theta^3\rangle \equiv \frac{\partial}{\partial x_3}\langle \theta^3\rangle, \end{aligned}\right\} \tag{2.1}$$

where $\boldsymbol{u} \equiv \boldsymbol{u}(\boldsymbol{x}, t)$ is the velocity fluctuation at position $\boldsymbol{x}$ and time t, $\theta \equiv \theta(\boldsymbol{x}, t)$ is the potential temperature fluctuation at $\boldsymbol{x}$ and t, subscript 3 denotes the vertical component, and the angular brackets denote an ensemble average. Our goal is to calculate these third moments in terms of second moments. We consider $\langle u_3 \boldsymbol{uu}\rangle$ first.

To calculate $\langle u_3 \boldsymbol{uu}\rangle$ we utilize the Navier–Stokes equation. The fluctuation part of that equation is given by

$$\left(\frac{\partial}{\partial t} + \nu\nabla^2\right)\boldsymbol{u} = -(\boldsymbol{u}\cdot\boldsymbol{\nabla u})' - \boldsymbol{u}\cdot\boldsymbol{\nabla U} - \boldsymbol{U}\cdot\boldsymbol{\nabla u} - \frac{\boldsymbol{\nabla} p}{\rho_0} - \frac{\boldsymbol{g}\theta}{\Theta_0}, \tag{2.2}$$

where p is the pressure fluctuation, $\boldsymbol{g}$ is the acceleration of gravity, ρ_0 is the mean particle density, and ν is the molecular viscosity. The molecular dissipation term has been placed on the left-hand side for later convenience, and we defined $(\boldsymbol{u}\cdot\boldsymbol{\nabla u})' \equiv (\boldsymbol{u}\cdot\boldsymbol{\nabla u}) - \langle \boldsymbol{u}\cdot\boldsymbol{\nabla u}\rangle$ for notational convenience.

Formal integration of this equation yields

$$\boldsymbol{u}(t) = G_\nu(t)\,\boldsymbol{u}(0) - \int_0^t \mathrm{d}t_1\, G_\nu(t-t_1)\,[(\boldsymbol{u}\cdot\boldsymbol{\nabla u})' + \boldsymbol{U}\cdot\boldsymbol{\nabla u} + \boldsymbol{I}(t_1)], \tag{2.3a}$$

$$\boldsymbol{I}(t_1) \equiv \frac{\boldsymbol{\nabla} p}{\rho_0} + \boldsymbol{u}\cdot\boldsymbol{\nabla U} + \frac{\boldsymbol{g}_\theta}{\Theta_0}, \tag{2.3b}$$

$$G_\nu(t) \equiv \exp(-\nu\nabla^2 t), \tag{2.4}$$

where it is understood that $\boldsymbol{u}$, p, θ, and $\boldsymbol{U}$ in the integrand are all evaluated at time t_1 (e.g. $\boldsymbol{u} \equiv \boldsymbol{u}(t_1)$ in the integrand of (2.3)), and we have separated $(\boldsymbol{u}\cdot\boldsymbol{\nabla u})' + \boldsymbol{U}\cdot\boldsymbol{\nabla u}$ from $\boldsymbol{I}(t_1)$ for later convenience. Equation (2.3) can be verified by differentiating both sides and comparing with (2.2) – as is shown in Appendix A. This equation will allow us to easily evaluate the influence of molecular dissipation in a rigorous fashion.

Substitution of the 3-component of (2.3) for u_3 in $\langle u_3\,\boldsymbol{uu}\rangle$ we obtain an equation for $\langle u_3\,\boldsymbol{uu}\rangle$. Two other equations for $\langle u_3\,\boldsymbol{uu}\rangle$ are obtained by substitution of (2.3) for the first and second $\boldsymbol{u}$ in $\langle u_3\,\boldsymbol{uu}\rangle$, alternately. The sum of all three equations is

$$3\langle u_3\,\boldsymbol{uu}\rangle = \mathrm{IV} - \int_0^t \mathrm{d}t_1 \Big[\boldsymbol{A}(t,t_1) + (1+\mathrm{T_r})\langle u_3(t)\,\boldsymbol{u}(t)\,G_\nu(t-t_1)\,\boldsymbol{I}(t_1)\rangle + \langle \boldsymbol{u}(t)\,\boldsymbol{u}(t)\,G_\nu(t-t_1)\,I_3(t_1)\rangle\Big], \tag{2.5}$$

$$\boldsymbol{A}(t,t_1) \equiv (1+\mathrm{T_r})\langle u_3(t)\,\boldsymbol{u}(t)\,G_\nu(t-t_1)\,[\boldsymbol{u}(t_1)\cdot\boldsymbol{\nabla}\,\boldsymbol{u}(t_1)]'\rangle + \langle \boldsymbol{u}(t)\,\boldsymbol{u}(t)\,G_\nu(t-t_1)\,[\boldsymbol{u}(t_1)\cdot\boldsymbol{\nabla} u_3(t_1)]'\rangle \tag{2.6}$$

when $\mathrm{T_r}$ denotes the transpose of a tensor [e.g. $\mathrm{T_r}\,\boldsymbol{ab} \equiv \boldsymbol{ba}$], IV denotes the initial-value term

$$[(1+\mathrm{T_r})\langle u_3(t)\,\boldsymbol{u}(t)\,G_\nu(t)\,\boldsymbol{u}(0)\rangle + \langle \boldsymbol{u}(t)\,\boldsymbol{u}(t)\,G_\nu(t)\,u_3(0)\rangle],$$

and all the terms containing $\boldsymbol{U}\cdot\boldsymbol{\nabla}$ collectively vanish because their sum is of the form $\boldsymbol{U}\cdot\boldsymbol{\nabla}\langle u_3\,\boldsymbol{u}G_\nu\,\boldsymbol{u}\rangle$, a form that is zero since $\boldsymbol{U}$ is along the x_1 direction while $\langle u_3\,\boldsymbol{u}G_\nu\,\boldsymbol{u}\rangle$ varies only with x_3. The initial-value term is henceforth neglected since,when divided by $\langle u_3(t)\,\boldsymbol{u}(t)\,\boldsymbol{u}(t)\rangle$, it decays towards zero as t increases. The timescale for this decay is, in the absence of buoyancy or viscosity, the Eulerian timescale $(k_0 v_0)^{-1}$ (e.g. Weinstock 1981), where $k_0 = 2\pi/L_0$ is a wavenumber characteristic of the energy-containing scales and v_0 is the r.m.s. velocity fluctuation – and a large $tk_0 v_0$ limit in (2.5) conforms to our basic assumption of small variations on a Lagrangian (or Eulerian) timescale. Equation (2.5) determines $\langle u_3\,\boldsymbol{uu}\rangle$ in terms of the two-time third and fourth moments $\boldsymbol{A}(t,t_1)$, $\langle \boldsymbol{uu}G_\nu\,\boldsymbol{\nabla}p\rangle$, and $\langle u_3\,\boldsymbol{u}G_\nu\,\theta\rangle$. This equation does not occur in the eddy-damping approach, nor do two-time moments.

To achieve closure of (2.5), the two-time moments on the right-hand side must be expressed in terms of single-time second moments. We consider $\boldsymbol{A}(t,t_1)$ first. An expression for

$$\int_0^t \mathrm{d}t_1\,\boldsymbol{A}(t,t_1),$$

based on a two-point closure and cumulant expansion, is derived in Appendix B §B.1 and given as follows:

$$\int_0^t \mathrm{d}t_1 \boldsymbol{A}(t,t_1) = \tau_0 \boldsymbol{A}^0 + \int_0^t \mathrm{d}t_1 \boldsymbol{Q}^{(4)}(t-t_1), \tag{2.7}$$

$$\boldsymbol{A}^0 = \langle u_3^2 \rangle \frac{\partial}{\partial x_3} \langle \boldsymbol{u}\boldsymbol{u} \rangle + (1+\mathrm{T_r}) \langle u_3 \boldsymbol{u} \rangle \frac{\partial}{\partial x_3} \langle u_3 \boldsymbol{u} \rangle, \tag{2.8}$$

$$\tau_0 \approx \frac{(\Pi^{\frac{1}{2}}/4)\,(6\epsilon/q^2)\,(1+R_\nu^{-1})}{(\Pi/2)N^2 H(N^2) + (6\epsilon/q^2)^2\,(1+R_\nu^{-1})^2}, \tag{2.9}$$

where R_ν is a Reynolds number

$$R_\nu \equiv (4/\Pi)^{\frac{1}{2}} v_0/\nu k_0).$$

$\boldsymbol{Q}^4(t-t_1)$ is the cumulant of $\boldsymbol{A}(t,t_1)$ – a two-time fourth-order cumulant, ϵ is the dissipation rate of kinetic energy density, $q^2 \equiv \langle \boldsymbol{u}\cdot\boldsymbol{u} \rangle$ is twice the kinetic energy density, $N \equiv (g\Theta_0^{-1}\partial\Theta_0/\partial x_3)^{\frac{1}{2}}$ is the Brunt–Väisälä frequency, $\langle \boldsymbol{u}\boldsymbol{u} \rangle \equiv \langle \boldsymbol{u}(t)\,\boldsymbol{u}(t) \rangle$ is the single-time second moment, τ_0 is the correlation time (decay time) of $\boldsymbol{A}(t,t_1)$, and H is the Heaviside step function ($H = 1$ when $N^2 \geqslant 0$, and $H = 0$ when $N^2 < 1$). The decay time τ_0 includes the influence of stable buoyancy and viscosity on turbulence decay (e.g. Weinstock 1978; Lumley *et al.* 1978, §3) and the H-function reflects the fact that unstable buoyancy does not influence the correlation time in an explicit manner. This influence of molecular viscosity ν comes from the term $G_\nu(t-t_1)$ and is very small when the Reynolds number R_ν is large.

In a similar fashion it is shown in Part 2 of Appendix B that, to first order in mean field gradients, the integrals of the other two-time functions in (2.5) can be expressed in terms of single-time functions given by

$$\int_0^t \mathrm{d}t_1 \langle u_3(t)\,\boldsymbol{u}(t)\,G_\nu(t-t_1)\,\boldsymbol{\nabla}p(t_1) \rangle \approx \tau_0 \langle u_3 \boldsymbol{u}\boldsymbol{\nabla}p \rangle;$$

$$\int_0^t \mathrm{d}t_1 \langle u_3(t)\,\boldsymbol{u}(t)\,G_\nu(t-t_1)\,\boldsymbol{u}(t_1) \rangle \approx \tau_0 \langle u_3 \boldsymbol{u}\boldsymbol{u} \rangle;$$

$$\int_0^t \mathrm{d}t_1 \langle u_3(t)\,\boldsymbol{u}(t)\,G_\nu(t-t_1)\,\theta(t_1) \rangle \approx \tau_0 \langle \boldsymbol{u}_3 u\theta \rangle;$$

etc. provided that mean values vary slowly on timescales τ_0, and t is much larger than τ_0. Substitution of these relations into (2.5), and neglecting the initial-value term when t is large, our expression for $\langle u_3 \boldsymbol{u}\boldsymbol{u} \rangle$ becomes

$$3\langle u_3 \boldsymbol{u}\boldsymbol{u} \rangle = -\tau_0 \left[\boldsymbol{A}^0 + (1+\mathrm{T_r}) \frac{\langle u_3 \boldsymbol{u}\boldsymbol{\nabla}p \rangle}{\rho_0} + \left\langle \rho_0^{-1} \boldsymbol{u}\boldsymbol{u} \frac{\partial p}{\partial x_3} \right\rangle + \langle \boldsymbol{u}\boldsymbol{u}\theta \rangle \frac{\boldsymbol{g}_3}{\Theta_0} \right.$$
$$\left. + (1+\mathrm{T_r}) \left(\langle u_3 \boldsymbol{u}\boldsymbol{u} \rangle \cdot \boldsymbol{\nabla}\boldsymbol{U} + \langle u_3 \boldsymbol{u}\theta \rangle \frac{\boldsymbol{g}}{\Theta_0} \right) \right] - \int_0^t \mathrm{d}t_1 \boldsymbol{Q}^{(4)}(t-t_1). \tag{2.10}$$

The basic approximation we make in this equation is to neglect the fourth-order cumulant term

$$\int_0^t \mathrm{d}t_1 \boldsymbol{Q}^{(4)}(t-t_1)$$

in comparison with $\tau_0 \boldsymbol{A}^0$. This neglect, suggested by Chandrasekhar (1955) and analysed by Kraichnan (1957), might be weaker than neglect of the single-term

fourth-order cumulant $\boldsymbol{Q}^{(4)}(0)$ in quasi-normal theory (e.g. Proudman & Reid 1954; Lumley *et al.* 1987) since $\boldsymbol{Q}^{(4)}(t-t_1)$ is a two-time cumulant and decays as $t-t_1$ increases. On the one hand, neglect of $\boldsymbol{Q}^{(4)}(t-t_1)$ does not violate realizability as does neglect of $\boldsymbol{Q}^{(4)}(0)$ in quasi-normal theory, while, on the other hand, as pointed out by a reviewer, $\boldsymbol{Q}^{(4)}(t-t_1)$ neglect is, itself, flawed since it causes non-conservative energy transfer by nonlinear interactions (Kraichnan 1957). Nevertheless, although inexact, there is evidence that this neglect gives reasonable results since it is basic to contemporary closure theories such as the direct interaction approximation (Kraichnan 1959) which gives reasonable results – particularly for the energy-containing scales. A correction for non-negligible $\boldsymbol{Q}^{(4)}$ is suggested in §5.

To evaluate the pressure–stress correlations in (2.10) we use the well-known incompressibility expression for p obtained by taking the divergence of (2.2):

$$\left.\begin{aligned} p &= p^{(\mathrm{s})}+p^{(U)}+p^{(\theta)}, \\ \nabla^2 p^{(\mathrm{s})} \equiv -\boldsymbol{\nabla}\cdot(\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u})', &\quad \nabla^2 p^{(U)} \equiv -2\boldsymbol{\nabla}\boldsymbol{u}:\boldsymbol{\nabla}\boldsymbol{U}, \quad \nabla^2 p^{(\theta)} \equiv -\boldsymbol{g}\cdot\boldsymbol{\nabla}\frac{\theta}{\Theta_0}, \end{aligned}\right\} \tag{2.11}$$

where $p^{(\mathrm{s})}$ is referred to as the slow part of p, and $p^{(U)}$ and $p^{(\theta)}$ are referred to as fast parts since they respond immediately to changes of $\boldsymbol{U}$ or Θ_0, respectively. Substituting (2.11) into the pressure correlations of (2.10) we obtain the three sets of terms corresponding to the slow and fast parts of p:

$$\left.\begin{aligned} (1+\mathrm{T_r})\frac{\langle u_3\boldsymbol{u}\boldsymbol{\nabla}p\rangle}{\rho_0}+\frac{1}{\rho_0}\left\langle \boldsymbol{u}\boldsymbol{u}\frac{\partial p}{\partial x_3}\right\rangle &\equiv \boldsymbol{\Pi}^{(\mathrm{s})}+\boldsymbol{\Pi}^{(U)}+\boldsymbol{\Pi}^{(\theta)}, \\ \boldsymbol{\Pi}^{(q)} \equiv (1+\mathrm{T_r})\rho_0^{-1}\langle u_3\boldsymbol{u}\boldsymbol{\nabla}p^{(q)}\rangle+\rho_0^{-1}&\left\langle \boldsymbol{u}\boldsymbol{u}\frac{\partial p^{(q)}}{\partial x_3}\right\rangle, \quad q=\mathrm{s}, \quad U, \quad \text{or } \theta. \end{aligned}\right\} \tag{2.12}$$

The slow terms are calculated in Appendix D in the limit of weak inhomogeneity and weak anisotropy, and are found to be given by

$$\boldsymbol{\Pi}^{(\mathrm{s})} = \boldsymbol{f}\cdot\boldsymbol{A}^0, \quad |f_{ij}| \ll 1, \tag{2.13}$$

where f_{ij} are numerical constants whose magnitudes are all much less than unity. In fact $|f_{ij}| \leqslant 0.1$. Consequently, the slow pressure term is much smaller than $\boldsymbol{A}^0$ and can be safely neglected in (2.10) – at least for the case of weak inhomogeneity. This smallness of $\boldsymbol{\Pi}^{(\mathrm{s})}$ contradicts the eddy-damping approximation. A discussion of eddy damping and its comparison with (2.10) and (2.13) is given in §§3 and 6.

Turning our attention to the rapid buoyancy part of (2.12), this part was previously obtained by Lumley *et al.* (1978). (We verified their expression in an approximate way by a cumulant-neglect calculation.) The expression is quite close to

$$\boldsymbol{\Pi}^{(\theta)} \approx -\frac{\tau_0}{5}(1+\mathrm{T_r})\frac{\langle u_3\boldsymbol{u}\theta\rangle\boldsymbol{g}}{\Theta_0}-\frac{\tau_0}{5}\frac{\langle \boldsymbol{u}\boldsymbol{u}\theta\rangle g_3}{\Theta_0}. \tag{2.14a}$$

With regard to the rapid shear term $\boldsymbol{\Pi}^{(U)}$, it can be formally expressed as (see Part 1 of Appendix E)

$$\boldsymbol{\Pi}^{(U)} = -2\frac{\partial U_0}{\partial x_3}\int\frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3}\left[(1+\mathrm{T_r})\langle u_3\boldsymbol{u}u_3^\dagger(\boldsymbol{k})\rangle\frac{\boldsymbol{k}k_1}{k^2}+\langle \boldsymbol{u}\boldsymbol{u}u_3^\dagger(\boldsymbol{k})\rangle\frac{k_1k_3}{k^2}\right]\exp(i\boldsymbol{k}\cdot\boldsymbol{x}),$$

where $u_3^\dagger(\boldsymbol{k})$ denotes the Fourier transform of $u_3(x)$ at Fourier wavevector $\boldsymbol{k}$ (the undaggered quantities u_3 and u_1 are not Fourier transformed). This $\boldsymbol{k}$-integral is too

difficult to evaluate, except in the asymptotic limit of small mean gradient magnitudes – small $|\tau_0 \partial U_0/\partial x_3|$ and $|\tau_0^2 g\theta_0^{-1} \partial\theta_0/\partial x_3|$ – and small anisotropy. In that limit it is shown in Appendix D that

$$\boldsymbol{\Pi}^{(U)} = -\tfrac{2}{5}\langle u_3^3\rangle (\boldsymbol{\nabla U} + \boldsymbol{\nabla U}^{\mathrm{T}}) + \boldsymbol{\delta}_0, \tag{2.14b}$$

where $\boldsymbol{\nabla U}^{\mathrm{T}}$ denotes the transpose of $\boldsymbol{\nabla U}$, and δ_0 denotes terms of order $(\partial U_0/\partial x_3)^2$ and $(\partial U_0/\partial x_3)(\partial\theta_0/\partial x_3)$. We do not know how $|\Pi_{ij}^{(U)}|$ will vary with increasing anisotropy or increasing $(\tau_0 \partial U_0/\partial x_3)^2$; whether or not it will increase. There is some evidence that $|\Pi_{33}^{(U)}|$ may be large for strong shears in the planetary boundary layer (Wyngaard 1979, §5) and a crude theoretical estimate of $\boldsymbol{\delta}_0$, applicable when stratification is negligible, is $\boldsymbol{\delta}_0 \sim -\tau_0 (\partial U_0/\partial x_3)^2 \langle u_3^3\rangle [\boldsymbol{I} - (4/5)\hat{\boldsymbol{x}}_2 \hat{\boldsymbol{x}}_2]$, where $\boldsymbol{I}$ denotes the unit dyadic. Consequently, we use (2.14*b*) but keep in mind that it may be very poor when $(\tau_0 \partial U_0/\partial x_3)^2$ is not small.

Finally, substitution of (2.12) and (2.14*a*, *b*) in (2.10) and neglecting $\boldsymbol{Q}^{(4)}$ – our basic cumulant-neglect approximation – yields the flux $\langle u_3 \boldsymbol{uu}\rangle$ in the desired form:

$$3\langle u_3 \boldsymbol{uu}\rangle = -\tau_0 \Big[\boldsymbol{A}^0 + \tfrac{4}{5}(1+\mathrm{T_r})\langle u_3 \boldsymbol{u}\theta\rangle \frac{\boldsymbol{g}}{\Theta_0} + \tfrac{4}{5}\langle \boldsymbol{uu}\theta\rangle \frac{g_3}{\Theta_0} + (1+\mathrm{T_r})(\langle u_3 \boldsymbol{uu}\rangle \cdot \boldsymbol{\nabla U} - \tfrac{2}{5}\langle u_3^3\rangle \boldsymbol{\nabla U}) \Big], \tag{2.15}$$

neglecting $\boldsymbol{\delta}_0$. This equation determines $\langle u_3 \boldsymbol{uu}\rangle$ in terms of $\langle \boldsymbol{uu}\theta\rangle$ and the second-moment term $\boldsymbol{A}^0$. Although derived from different considerations, (2.15) is similar to equation (18) obtained by Lumley *et al.* (1978) in their application of the eddy-damping quasi-normal approximation. One difference is that molecular viscosity occurs only in the relaxation time τ_0, a formulation of viscosity which simplifies the analysis of fluxes at the end of this section. Minor differences are that (2.15) includes $\boldsymbol{\nabla U}$ and that τ_0 depends on N^2. The latter dependence causes a stronger influence of stratification on fluxes. However, the difference of most interest to us – between (2.15) and (18) of Lumley *et al.* or (A 5) of Hanjalić & Launder (1972) – is in the underlying approximations of the respective derivations. Comparison of these approximations is made in §§3 and 6.

To close (2.15), $\langle \boldsymbol{uu}\theta\rangle$ must be expressed in terms of second moments. Such an equation is derived by substitution of the thermodynamic equation (potential temperature equation) for θ into $\langle \boldsymbol{uu}\theta\rangle$, and, in addition, substitution of (2.3) for $\boldsymbol{u}$. The derivation is entirely similar to the derivation of (2.10). The details of this derivation are given in Part 1 of Appendix F, together with outlined, similar derivations of $\langle \boldsymbol{u}\theta\theta\rangle$ and $\langle\theta^3\rangle$. The results for the case $t \gg \tau_0$ are

$$3\langle \boldsymbol{uu}\theta\rangle = -\tau_0 \Big[\boldsymbol{A}^{(\theta)} + (1+\mathrm{T_r})\frac{1}{\rho_0}\Big(\langle \boldsymbol{u}\nabla p\theta\rangle + \langle \boldsymbol{u}\theta\theta\rangle \frac{\boldsymbol{g}}{\Theta_0} + \langle \theta \boldsymbol{uu}\rangle \cdot \boldsymbol{\nabla U} \Big) + \langle \boldsymbol{uuu}\rangle \cdot \boldsymbol{\nabla}\Theta_0 \Big] - \int_0^t \mathrm{d}t_1 \, \boldsymbol{Q}^{(\theta)}(t-t_1), \tag{2.16}$$

$$3\langle \boldsymbol{u}\theta\theta\rangle = -\tau_0 \Big[\boldsymbol{A}^{(\theta^2)} + \frac{1}{\rho_0}\langle \nabla p\theta^2\rangle + \langle\theta^3\rangle \frac{\boldsymbol{g}}{\Theta_0} + 2\langle \theta \boldsymbol{uu}\rangle \cdot \boldsymbol{\nabla}\Theta_0 + \langle \theta^2 \boldsymbol{u}\rangle \cdot \boldsymbol{\nabla U} \Big] - \int_0^t \mathrm{d}t_1 \, \boldsymbol{Q}^{(\theta^2)}(t-t_1), \tag{2.17}$$

$$\langle\theta^3\rangle = -\tau_0 [A^{(\theta^3)} + \langle \theta^2 \boldsymbol{u}\rangle \cdot \boldsymbol{\nabla}\Theta_0] - \int_0^t \mathrm{d}t_1 \, Q^{(\theta^3)}(t-t_1), \tag{2.18}$$

$$\langle \boldsymbol{A}^{(\theta)} \rangle = \langle u_3 \theta \rangle \frac{\partial}{\partial x_3} \langle \boldsymbol{u}\boldsymbol{u} \rangle + (1 + \mathrm{T_r}) \langle u_3 \boldsymbol{u} \rangle \frac{\partial}{\partial x_3} \langle \boldsymbol{u}\theta \rangle, \tag{2.19}$$

$$\langle \boldsymbol{A}^{(\theta^2)} \rangle = \langle u_3 \boldsymbol{u} \rangle \frac{\partial}{\partial x_3} \langle \theta^2 \rangle + 2 \langle u_3 \theta \rangle \frac{\partial}{\partial x_3} \langle \boldsymbol{u}\theta \rangle, \tag{2.20}$$

$$\langle A^{(\theta^3)} \rangle = \langle u_3 \theta \rangle \frac{\partial}{\partial x_3} \langle \theta^2 \rangle, \tag{2.21}$$

where $\boldsymbol{Q}^{(\theta)}(t-t_1)$, $\boldsymbol{Q}^{(\theta^2)}(t-t_1)$, and $Q^{(\theta^3)}(t-t_1)$ are fourth-order cumulants of

$$(2\langle \boldsymbol{u}(t)\,\theta(t)\,[\boldsymbol{u}(t_1)\cdot\boldsymbol{\nabla}\boldsymbol{u}(t_1)]'\rangle + \langle \boldsymbol{u}(t)\,\boldsymbol{u}(t)\,[\boldsymbol{u}(t_1)\cdot\boldsymbol{\nabla}\theta(t_1)]'\rangle),$$

$$(2\langle \boldsymbol{u}(t)\,\theta(t)\,[\boldsymbol{u}(t_1)\cdot\boldsymbol{\nabla}\theta(t_1)]'\rangle + \langle \theta(t)\,\theta(t)\,[\boldsymbol{u}(t_1)\cdot\boldsymbol{\nabla}\boldsymbol{u}(t_1)]'\rangle),$$

and

$$\langle \theta^2(t)\,[\boldsymbol{u}(t_1)\cdot\boldsymbol{\nabla}\theta(t_1)]'\rangle,$$

respectively. Our basic approximation is to neglect these two-time fourth-order cumulants in (2.16), (2.17) and (2.18). Implications of our approximations are considered in §4.

Each pressure term in (2.16) and (2.17) has three parts – $p^{(\mathrm{s})}$, $p^{(U)}$, and $p^{(\theta)}$, the slow, rapid mean shear, and rapid buoyancy parts of p given by (2.11). The slow parts of the pressure terms are calculated in Part 2 of Appendix F. They are given by

$$(1 + \mathrm{T_r}) \frac{1}{\rho_0} \langle \boldsymbol{u}\boldsymbol{\nabla} p^{(\mathrm{s})} \theta \rangle = \boldsymbol{f}\cdot\boldsymbol{A}^{(\theta)}, \tag{2.22a}$$

$$\frac{1}{\rho_0} \langle \theta^2 \boldsymbol{\nabla} p^{(\mathrm{s})} \rangle = \hat{\boldsymbol{x}}_3 \cdot \boldsymbol{f}\cdot\boldsymbol{A}^{(\theta^2)} \tag{2.22b}$$

where the f_{ij}, the same as in (2.13), have very small magnitudes (less than 0.1) and both sides of these equations are evaluated for very small anisotropy. These $\boldsymbol{\nabla}_p^{(\mathrm{s})}$ correlations are negligibly small – in contradiction to the eddy-damping assumption.

The rapid mean shear terms are calculated in Part 2 of Appendix F for the case of asymptotically small mean gradients and anisotropy. There it is found that

$$(1 + \mathrm{T_r}) \frac{1}{\rho_0} \langle \boldsymbol{u}\boldsymbol{\nabla} p^{(U)} \theta \rangle \sim -\tfrac{3}{5} \langle u_3^2 \theta \rangle (\boldsymbol{\nabla}\boldsymbol{U} + \boldsymbol{\nabla}\boldsymbol{U}^{\mathrm{T}}), \tag{2.23a}$$

$$\frac{1}{\rho_0} \langle \boldsymbol{\nabla} p^{(U)} \theta^2 \rangle \sim -\tfrac{4}{5} \langle \theta^2 \boldsymbol{u} \rangle \cdot \boldsymbol{\nabla}\boldsymbol{U}. \tag{2.23b}$$

The rapid buoyancy terms were calculated by Lumley *et al.* (1978), approximately. In vector notation, their equations (16) and (8) are

$$(1 + \mathrm{T_r}) \frac{1}{\rho_0} \langle \boldsymbol{u}\boldsymbol{\nabla} p^{(\theta)} \theta \rangle = -\tfrac{3}{10}(1 + \mathrm{T_r}) \langle \boldsymbol{u}\theta^2 \rangle \frac{\boldsymbol{g}}{\Theta_0} + \tfrac{1}{5} \boldsymbol{I} \langle u_3 \theta^2 \rangle \frac{g_3}{\Theta_0}, \tag{2.24}$$

$$\langle \boldsymbol{\nabla} p^{(\theta)} \theta^2 \rangle = \tfrac{1}{3} \langle \theta^3 \rangle \frac{\boldsymbol{g}}{\Theta_0}, \tag{2.25}$$

where $\boldsymbol{I}$ is the identity matrix ($I_{ik} \equiv \delta_{ik}$).

Substituting (2.22)–(2.25) into (2.16) and (2.17), and neglecting two-time fourth-order cumulants, we obtained the desired expressions:

$$3\langle \boldsymbol{uu}\theta\rangle = -\tau_0\Big[\boldsymbol{A}^{(\theta)} + \tfrac{7}{10}(1+\mathrm{T_r})\langle \boldsymbol{u}\theta^2\rangle \frac{\boldsymbol{g}}{\Theta_0} + \frac{\boldsymbol{I}}{5}\langle u_3\theta^2\rangle \frac{g_3}{\Theta_0} + \langle \boldsymbol{uuu}\rangle\cdot\nabla\Theta_0 + (1+\mathrm{T_r})(\langle \theta\boldsymbol{uu}\rangle\cdot\nabla\boldsymbol{U} - \tfrac{2}{5}\langle u_3^2\theta\rangle\nabla\boldsymbol{U})\Big], \quad (2.26)$$

$$3\langle \boldsymbol{u}\theta^2\rangle = -\tau_0\Big[\boldsymbol{A}^{(\theta^2)} + \tfrac{2}{3}\langle\theta^3\rangle\frac{\boldsymbol{g}}{\Theta_0} + 2\langle\theta\boldsymbol{uu}\rangle\cdot\nabla\Theta_0 + \tfrac{1}{5}\langle\theta^2\boldsymbol{u}\rangle\cdot\nabla\boldsymbol{U}\Big], \quad (2.27)$$

$$\langle\theta^3\rangle = -\tau_0[\boldsymbol{A}^{(\theta^3)} + \langle\theta^2\boldsymbol{u}\rangle\cdot\nabla\Theta_0]. \quad (2.28)$$

Equations (2.26), (2.27), and (2.28) together with (2.15), constitute a closure; i.e. this set of equations determines the third moments in terms of second moments (the second moments are contained in $\boldsymbol{A}$, $\boldsymbol{A}^{(\theta)}$, $\boldsymbol{A}^{(\theta^2)}$ and $A^{(\theta^3)}$). A similar set of equations was first derived by Lumley *et al.* (1978). The differences are in the dissipation terms, relaxation time (denoted here by τ_0) and inclusion of a mean shear aside from the method of derivation – as discussed in the next paragraph. A substantive disagreement occurs only in the respective equations for $\langle\theta^3\rangle$.

All that remains is to solve this set of equations for the individual fluxes $\langle u_3\boldsymbol{uu}\rangle$, $\langle\boldsymbol{uu}\theta\rangle$, and $\langle\boldsymbol{u}\theta^2\rangle$ in terms of second moments – a straightforward task since the equations are linear in the fluxes. The solution is conveniently displayed in the matrix format used originally by Lumley *et al.* (1978):

$$\begin{bmatrix}\langle u_3^3\rangle\\ \langle u_3u_1^2\rangle\\ \langle u_3^2\theta\rangle\\ \langle u_3\theta^2\rangle\end{bmatrix} = -\tau_0\begin{bmatrix}A_1 & 0 & 0 & A_2 & 0 & A_3\\ A_4 & A_5 & A_6 & 0 & A_7 & 0\\ A_8 & 0 & 0 & A_9 & 0 & A_{10}\\ A_{11} & 0 & 0 & A_{12} & 0 & A_{13}\end{bmatrix}\begin{bmatrix}\langle u_3^2\rangle'\\ \langle u_1^2\rangle'\\ \langle u_1u_3\rangle'\\ \langle u_3\theta\rangle'\\ \langle u_1\theta\rangle'\\ \langle\theta^2\rangle'\end{bmatrix}, \quad (2.29)$$

where the coefficients A_1–A_{13} are written out in Appendix C and, for simplicity, terms of order $(g/\Theta_0)^2$ and $(g/\Theta_0)\,(\partial U_0/\partial x_3)^2$ are neglected in A_4–A_7 – the coefficients of $\langle u_3u_1^2\rangle$. These expressions for A_1–A_{13} are put in an Appendix rather than given here in order to avoid distraction from our additional purpose. This purpose is to compare our method of derivation with the EDQN methods of Hanjalić & Launder (1972) and Lumley *et al.* (1978), and to use the comparison to help assess the eddy-damping assumption. Before turning our attention to this purpose we note that, as previously mentioned, there are also differences in the dimensionless numerical coefficients, which is to be expected since our coefficients are determined in an independent fashion – as integrals over products of spectra. In particular, the two principal coefficients are $\frac{1}{3}\tau_0(0)$ (Lumley *et al.* refer to it as τ_3) and the coefficient of terms first order in g/Θ_0. We obtain $\frac{1}{3}\tau_0(0) = 0.05\ q^2/\varepsilon$, where $q_2 \equiv \langle\boldsymbol{u}\cdot\boldsymbol{u}\rangle$, while Lumley *et al.* obtain $\frac{1}{3}\tau_0(0) \equiv \tau_3 \approx 0.1\ q^2/\varepsilon$. An empirical determination by Hanjalić & Launder (1972) gives $\frac{1}{3}\tau_0(0) \approx 0.04\ q^2/\varepsilon$ (more precisely Hanjalić & Launder give $\frac{1}{3}\tau_0 = 0.08\ E/\varepsilon$ where $E \equiv \frac{1}{2}q^2$). These values of τ_0 all lie within a factor of about two of each other. The differences in τ_0 are small considering the different, independent methods used. As for the coefficients of leading g/Θ_0 terms, ours exceeds theirs by a factor of about three for stable stratification, but is comparable to theirs for unstable stratification – with the difference for stable stratification due to the suppression of autocorrelation decay time τ_0 by stable buoyancy as described in (2.9).

Let us next compare the two different derivations of transport terms and focus our attention on the eddy-damping approximation and its consequences.

3. Validity of eddy-damping approximation

Equation (2.13) contradicts the eddy-damping approximation for single-point moments since it shows $|\Pi_{ij}^{(s)}|$ to be negligibly small, in the limit of weak inhomogeneity at least. The formal connection between (2.13) and the eddy-damping approximation may need clarification because that approximation is not usually expressed in terms of $\boldsymbol{A}^0$, but rather is stated in the form

$$\boldsymbol{\Pi}^{(s)} = c_s^{-1} \frac{\epsilon}{E} \langle u_3 \boldsymbol{u}\boldsymbol{u} \rangle \quad \text{(eddy-damping approximation)}, \tag{3.1}$$

where c_s is a proportionality constant empirically determined to be about 0.08 and $E \equiv \frac{1}{2} q^2$ is the turbulence kinetic energy density.

Let us consider the connection between this relation and the inequality (2.13) for the case of zero θ and negligible mean velocity terms – the case considered by Hanjalić & Launder (1972). In that case (2.15) reduced to

$$\frac{3}{2\tau_0} \langle u_3 \boldsymbol{u}\boldsymbol{u} \rangle = -\boldsymbol{A}^0 \quad \text{(theory)}. \tag{3.2}$$

The same expression is implicit in Hanjalić & Launder (see their Appendix). Now, it can be seen, by combination of (3.2) with inequality (2.13) and $2\tau_0/3 \approx c_s E/\epsilon$, that

$$|\Pi_{ij}^{(s)}| \ll |c_s^{-1} \frac{\epsilon}{E} \langle u_3 u_i u_j \rangle| \quad \text{(theory)}. \tag{3.3}$$

This inequality was derived entirely by theory and disagrees, formally, with the eddy-damping approximation – for weakly inhomogeneous turbulence.

Since $\boldsymbol{\Pi}^{(s)}$ is so relatively small there appears to be no need at all for the eddy-damping approximation of single-point triple moments – the case considered here. The questions that remain are why the EDQN equation is comparable with our (2.15) when the former relies on eddy damping whereas the latter does not, and why eddy damping is very useful for two-point moments (the kind of moments considered by Orszag 1970; Cambon, Jaendel & Mathieu 1981) but not for single-point moments. These questions are discussed in §6. First we point out a non-trivial consequence of (2.13) or (3.3) for dissipation.

4. The magnitude of dissipation

Thus far, we have not made use of the transport equation for $\langle u_3 \boldsymbol{u}\boldsymbol{u} \rangle$ – a fundamental equation and the starting place for the EDQN method. This equation should supply us with additional information since it includes short scales (dissipation) as well as larger scales, whereas (2.15) is biased towards larger scales by virtue of the time integration in its derivation (the time integration in (2.5) brings down a correlation time on the order of $k^{-1}u_k^{-1}$ as shown in Appendix B. The

additional information is readily found by writing down the transport equation and (2.15) in the concise forms

$$\frac{\mathrm{D}\langle u_3 \boldsymbol{uu}\rangle}{\mathrm{D}t} = -[\boldsymbol{A}^0 + \boldsymbol{\Pi}^{(s)} + \boldsymbol{F}] - \boldsymbol{Q}^{(4)}(0) - \boldsymbol{D}_3, \tag{4.1}$$

$$\frac{3}{\tau_0}\langle u_3 \boldsymbol{uu}\rangle = -[\boldsymbol{A}^0 + \boldsymbol{\Pi}^{(s)} + \boldsymbol{F}] - \frac{1}{\tau_0}\int_0^t \mathrm{d}t_1\, \boldsymbol{Q}^{(4)}(t-t_1) \quad \text{(present theory)}, \tag{4.2}$$

where $\boldsymbol{D}_3 = \nu\langle \boldsymbol{uu}\nabla^2 u_3\rangle + 2\nu\langle u_3 \boldsymbol{u}\nabla^2 \boldsymbol{u}\rangle$ denotes the dissipation term of our third moment $\langle u_3 \boldsymbol{uu}\rangle$ and $\boldsymbol{F}$ denotes all the mean field terms including the fast parts of $\boldsymbol{\Pi}$; i.e. $\boldsymbol{F} \equiv (1+\mathrm{T_r})(\langle u_3 \boldsymbol{uu}\rangle \cdot \nabla \boldsymbol{U} + \langle u_3 \boldsymbol{u}\theta\rangle \boldsymbol{g}/\Theta_0) + \langle u_3 \boldsymbol{uu}\rangle g_3/\Theta_0 + \boldsymbol{\Pi}^{(U)} + \boldsymbol{\Pi}^{(\theta)}$. The transport equation (4.1) is formally exact, whereas (4.2) was derived under the restriction (approximation) that all mean quantities vary but slowly on an Eulerian integral timescale and that t exceeds τ_0.

In the limit of constant flux and use of (2.13) to neglect $\boldsymbol{\Pi}^{(s)}$ in comparison with $\boldsymbol{A}^0$, combination of (3.3) and (4.1) yields

$$\frac{3}{\tau_0}\langle u_3 \boldsymbol{uu}\rangle = \boldsymbol{D}_3 + \boldsymbol{Q}^{(4)}(0) - \frac{1}{\tau_0}\int_0^t \mathrm{d}t\, \boldsymbol{Q}^{(4)}(t-t_1), \tag{4.3}$$

which relates $\langle u_3 \boldsymbol{uu}\rangle$ to $\boldsymbol{D}_3$ and $\boldsymbol{Q}^{(4)}(t-t_1)$; and we emphasize that (4.3) is justified for constant flux, weak inhomogeneity and slow variation of mean quantities.

This equation provides a possible determination of the dissipation, a quantity that is not known accurately from experiment or theory. It all depends on the magnitude of $\boldsymbol{Q}^{(4)}(t-t_1)$.

If $\boldsymbol{Q}^{(4)}(t-t_1)$ is sufficiently smaller than $\boldsymbol{A}^0$ at all $t-t_1$, then $\boldsymbol{D}_3$ is determined by (4.3).With regard to the magnitude of $\boldsymbol{Q}^{(4)}(0)$, experimental evidence indicates that it is significantly smaller than $\boldsymbol{A}^0$ in the convective atmospheric boundary layer (Wyngaard 1979). In qualitative agreement with this experiment is a theoretical derivation by Lumley (1978) – albeit for a partly arbitrary moment generating function. These findings are not contradicted by Deardorff (1978). He found that the fourth-order cumulant $\boldsymbol{Q}^{(\theta)}(0)$ is not zero and makes an important contribution to scalar diffusion, but this finding does not conflict with our neglect of $\boldsymbol{Q}^{(4)}(0)$ since mean fluxes were not constant in his calculation. Additionally, $\boldsymbol{Q}^{(\theta)}(0)$ was higher order in the inhomogeneity, sufficiently small to be neglected by us; i.e.

$$Q_{33}^{(\theta)}(0)/A_{33}^{(\theta)}(0) = O(a^2/k_0^2), \tag{4.4}$$

where $a \equiv \frac{1}{2}\partial \ln\langle \boldsymbol{u}\cdot\boldsymbol{u}\rangle/\partial x_3$ is the reciprocal scale length of spatial inhomogeneity. A similar ordering was also found by Lumley (1978). The importance found by Deardorff for $\boldsymbol{Q}^{(\theta)}$ was to remove narrow 'spikes' of concentration that formed in a rapidly diffusing scalar source, a situation that is excluded by our constant-flux limitation. The importance of this finding for us is that the constant-flux-case theory can be defective when applied to rapidly varying fluxes.

To determine $\boldsymbol{D}_3$ we need

$$\int_0^t \mathrm{d}t_1\, \boldsymbol{Q}^{(4)}(t-t_1)$$

as well as $\boldsymbol{Q}^{(4)}(0)$ to be negligible. The fact that $\boldsymbol{Q}^{(4)}(0)$ is small does not guarantee that $\boldsymbol{Q}^{(4)}(t-t_1)$ is small for all $t-t_1$, particularly for $t-t_1$ on the order of a correlation time. For present purposes we merely consider the relative smallness of $|\boldsymbol{Q}^{(4)}(t-t_1)|$

a possibility to be proven or disproven at a later time. If $|\boldsymbol{Q}^{(4)}(t-t_1)|$ is small then (4.3) does determine the dissipation; it is given by

$$\boldsymbol{D}_3 = \frac{18\bar{\epsilon}}{q^2}\langle u_3 \boldsymbol{u}\boldsymbol{u}\rangle \quad (\text{small}\,|\boldsymbol{Q}^{(4)}(t-t_1)|), \tag{4.5}$$

where we used $\tau_0^{-1} \approx 6\bar{\epsilon}/q^2$ from (B 7) (Appendix B). If desired, the value of $\boldsymbol{D}_3$ can be expressed in terms of second moments by substitution of (2.15) for $\langle u_3 \boldsymbol{u}\boldsymbol{u}\rangle$.

This simple looking result for $\boldsymbol{D}_3$ is similar in form to the previous expression $D_{ij} = (2\bar{\epsilon}/3q^2)(\delta_{ij} < q^2 u_3\rangle + \delta_{3i}\langle q^2 u_j\rangle + \delta_{3j}\langle q^2 u_i\rangle)$ suggested by Zeman & Lumley (1976) from a different consideration. The principal difference is in the numerical coefficient. Our coefficient is about five times the previous (rough) estimate as can be seen by taking the trace of both expressions and comparing. Unfortunately, there is no available experiment, to our knowledge, that can verify this coefficient. Such an experiment would be invaluable to us; it would not only determine whether or not $\boldsymbol{D}_3$ is correctly given by (4.5) but, by inference, would also confirm or deny the validity of the basic $\boldsymbol{Q}^{(4)}(t-t_1)$ neglect.

A final note about (4.5) is that it bears a formal relationship to the well-known expression for kinetic energy dissipation rate $\bar{\epsilon} \equiv \nu\langle \boldsymbol{u}\cdot\nabla^2\boldsymbol{u}\rangle$:

$$\nu\langle \boldsymbol{u}\cdot\nabla^2\boldsymbol{u}\rangle = \frac{\bar{\epsilon}}{q^2}\langle \boldsymbol{u}\cdot\boldsymbol{u}\rangle. \tag{4.6}$$

If the trace is taken of both sides of (4.5), and use is made of the nearly isotropic approximation $\nu\langle \boldsymbol{u}\cdot\boldsymbol{u}\nabla^2 u_3\rangle \approx \nu\langle u_3 \boldsymbol{u}\cdot\nabla^2\boldsymbol{u}\rangle$ (e.g. Zeman & Lumley 1976) we obtain

$$\nu\langle u_3(\boldsymbol{u}\cdot\nabla^2\boldsymbol{u}\rangle = \frac{6\bar{\epsilon}}{q^2}\langle u_3(\boldsymbol{u}\cdot\boldsymbol{u})\rangle \quad (\text{small}\,|\boldsymbol{Q}^{(4)}(t-t_1)|) \tag{4.7}$$

which has the same form as (4.6), but with a greater coefficient. If valid, the greater coefficient suggests that third-moment spectra are relatively large at short scales; i.e. the ratio of short scales to large scales is larger for third moments than for second moments. Equation (4.6) has been interpreted to mean that the destruction of large scales is a source of shorter, dissipative scales – a cascade. Such an interpretation of (4.7) is suggested by its derivation in which it is obtained by equating the large-scale-dominated (4.2) to the shorter-scale-containing (4.1). However, we emphasize that the accuracy of (4.7) is not certain since $\boldsymbol{Q}^{(4)}(t-t_1)$ was assumed to be small. An experimental verification is required.

5. Fourth-cumulant model

Our basic assumption for the derived transport equations in §2, and for the dissipation in §4, is neglect of the fourth-cumulant integral

$$\int_0^t \mathrm{d}t_1\, \boldsymbol{Q}^{(4)}(t-t_1).$$

In the event that this term is not small, a model for it is suggested by (4.3): namely

$$\boldsymbol{Q}^{(4)}(0) - \tau_0^{-1}\int_0^t \mathrm{d}t_1\, \boldsymbol{Q}^{(4)}(t-t_1) = B\frac{3}{\tau_0}\langle u_3 \boldsymbol{u}\boldsymbol{u}\rangle, \tag{5.1}$$

where B is an undetermined dimensionless coefficient. Actually, this model is

analogous to the EDQNM one (Orszag 1970; see, particularly, equation (2.7) of Cambon *et al.* 1981) with B/τ_0 an eddy viscosity. The noteworthy difference is that they considered a fourth-order cumulant at two points in space whereas we consider a fourth-order cumulant at two points in time. The transport equation (2.15) could be corrected by first substituting (5.1) in (2.10). The corrected (2.15) is then obtained by replacement of $\langle u_3 \boldsymbol{uu}\rangle$ with $(1-B)\langle u_3 \boldsymbol{uu}\rangle$ – assuming that $\boldsymbol{Q}^{(4)}(0)$ is small for small inhomogeneity.

The corrected dissipation is given by substitution of (5.1) in (4.3) to yield

$$\boldsymbol{D}_3 = (1-B)\frac{3}{\tau_0}\langle u_3 \boldsymbol{uu}\rangle. \tag{5.2}$$

The coefficient B is the only missing quantity of the theory and, from this perspective, it is this coefficient that is determined by comparison of theory with experiment. However, we caution that this model $\boldsymbol{Q}^{(4)}$ pertains only to the case of constant flux or slowly varying flux. It does not pertain to rapidly varying fluxes (e.g. Deardorff 1978).

6. Comparison of two transport theories

Finally we consider the seeming contradiction that has emerged about the EDQN approximation. This contradiction is as follows: on the one hand, (2.13) implies that the eddy-damping approximation is invalid, since it overestimates the magnitude of $\boldsymbol{\Pi}^{(s)}$, while, on the other hand, the overall EDQN equation of Hanjalić & Launder (1972) is in near agreement with our (2.15) for the case of θ equal to zero. (We refer to their (2.3) with $c_s = 0.08$.) To resolve this contradiction, we briefly rederive the EDQN equation and our (2.15) and show that they agree. A brief derivation is obtained from the constant-flux limit of (4.1) given by

$$\boldsymbol{\Pi}^{(s)} + \boldsymbol{D}_3 + \boldsymbol{Q}^{(4)}(0) = -(\boldsymbol{A}^0 + \boldsymbol{F}). \tag{6.1}$$

The EDQN equation is derived from this by neglecting $\boldsymbol{Q}^{(4)}(0)$, and assuming that $c_s\, \boldsymbol{\Pi}^{(s)} = \epsilon E^{-1}\langle u_3 \boldsymbol{uu}\rangle$. An additional, underlying assumption is that $\boldsymbol{D}_3$ is smaller than $\boldsymbol{\Pi}^{(s)}$ in magnitude, i.e. that molecular damping is smaller than eddy damping (Hanjalić & Launder neglect $\boldsymbol{D}_3$ altogether) – otherwise, if molecular damping were larger, there would be little need for $\boldsymbol{\Pi}^{(s)}$. With these assumptions, (6.1) reduces to

$$c_s^{-1}\frac{\bar{\epsilon}}{q^2}\langle u_3 \boldsymbol{uu}\rangle = -[\boldsymbol{A}^0 + \boldsymbol{F}] \quad \text{(EDQN)}. \tag{6.2}$$

This is the EDQN closure for third moments. In comparison, the present theory is derived by neglecting $\int dt_1\, \boldsymbol{Q}^{(4)}$ and showing $\boldsymbol{\Pi}^{(s)}$ to be small so that (4.2) reduces to

$$\frac{3}{\tau_0}\langle u_3 \boldsymbol{uu}\rangle = -[\boldsymbol{A}^0 + \boldsymbol{F}] \quad \text{(present theory)}. \tag{6.3}$$

Since $c_s q^2/\epsilon \approx 3/\tau_0$, it is evident that the two methods give the same equation although one neglects $\boldsymbol{\Pi}^{(s)}$ and the other depends on $\boldsymbol{\Pi}^{(s)}$ as a major quantity. This contradiction can be resolved by consideration of the 'total' damping $\boldsymbol{D}_3 + \boldsymbol{\Pi}^{(s)}$. An expression for this quantity is given by substitution of (6.1) in (6.3):

$$\boldsymbol{\Pi}^{(s)} + \boldsymbol{D}_3 = \frac{3}{\tau_0}\langle u_3 \boldsymbol{uu}\rangle. \tag{6.4}$$

As long as (6.4) is satisfied, both methods will agree – as may be seen by substituting (6.4) into (4.1) and (4.2).

Both methods do yield (6.4) for the 'total' damping, but for different reasons: the EDQN method takes $\boldsymbol{D}_3$ to be negligible and assumes $\boldsymbol{\Pi}^{(s)}$ equals $c_s^{-1}\bar{\epsilon}q^{-2}\langle u_3\boldsymbol{uu}\rangle$, whereas we find the reverse for $\boldsymbol{D}_3$ and $\boldsymbol{\Pi}^{(s)}$. Hence, both methods will give the same equation (6.4) although one treats $\boldsymbol{\Pi}^{(s)}$ as a major quantity whereas the other treats (finds) $\boldsymbol{\Pi}^{(s)}$ to be negligible. Simply put, it is the 'total' damping that matters and both methods agree on this 'total'.

In sum, what we have sought to show is that the EDQN method is correct because it gives the total damping $\boldsymbol{\Pi}^{(s)}+\boldsymbol{D}_3$ correctly – although empirically. Where it appears to be wrong is that it attributes the total damping mainly to $\boldsymbol{\Pi}^{(s)}$ rather than to $\boldsymbol{D}_3$. However, we again caution that this conclusion is based on the assumed negligibility of fourth-order cumulants.

Our conclusion, that molecular damping exceeds eddy damping, does not apply to the Fourier-space (two-point) case originally considered by Orszag (1970) since molecular damping was extremely small for the pertinent scales sizes – energy-containing and (much of the) inertial-subrange scales. The dominance of molecular over eddy damping is more tenable for the single-point-moments quantities considered by us since they include all scales including dissipation scales. In fact, one could argue that eddy damping should not exceed molecular dissipation since the net loss for single-point moments is by molecular dissipation. Eddy damping is not needed.

7. Summary and conclusions

Turbulence transport terms (third-moments) were calculated by a closure method that avoids the eddy-damping assumption. The approximation we do make is to neglect

$$\tau_0^{-1}\int_0^t \mathrm{d}t\,\boldsymbol{Q}^{(4)}(t-t_1)$$

in comparison with $\boldsymbol{A}^0$. The calculation is also limited to small spatial inhomogeneities, slow temporal variations on a Lagrangian timescale, and small anisotropy for evaluating $\boldsymbol{\Pi}$. Our conclusions are:

(*a*) It is proven that $|\Pi_{ij}^{(s)}| \ll |A_{ij}^0|$, $|\Pi_{ij}^{(s)}| \ll (3/\tau_0)|\langle u_3 u_i u_j\rangle|$ in the limit of small inhomogeneity and small anisotropy. Consequently, eddy damping is not valid for third-moment quantities in real space; it is not even needed.

(*b*) Molecular damping (dissipation) exceeds eddy damping for single-point moments. The relative values of these dampings are the reverse of the Fourier space case originally considered by Orszag (1970). That case pertained to relatively large scale sizes for which molecular damping is very small – energy-containing and larger scales of the inertial subrange. The dominance of molecular over eddy damping is understandable for single-point, single-time quantities since these quantities include dissipation scales. Physically, molecular damping should dominate since it is the net loss of real space fluxes.

(*c*) The dissipation rate of third moments of velocity is derived to be

$$\boldsymbol{D}_3 = 18(\bar{\epsilon}/q^2)\langle u_3\boldsymbol{uu}\rangle,$$

provided that $\boldsymbol{Q}^{(4)}$ is negligibly small. The coefficient is 5 times larger than a previous estimate (e.g. Zeman & Lumley 1976).

(*d*) The agreement between EDQN and our theory is attributed to a cancellation of errors regarding the 'total' dissipation rate $\boldsymbol{\Pi}^{(s)}+\boldsymbol{D}_3$. Both methods give $\boldsymbol{\Pi}^{(s)}+\boldsymbol{D}_3=(3/\tau_0)\langle u_3\boldsymbol{uu}\rangle$. The difference is that the EDQN method assumes $|\boldsymbol{D}_3|\ll|\boldsymbol{\Pi}^{(s)}|$ and $\boldsymbol{\Pi}^{(s)}=(3/\tau_0)\langle u_3\boldsymbol{uu}\rangle$ (eddy damping), whereas the theory shows the reverse; i.e. $|\boldsymbol{\Pi}^{(s)}|\ll|\boldsymbol{D}|$, and $\boldsymbol{D}_3=(3/\tau_0)\langle u_3\boldsymbol{uu}\rangle$.

(*e*) In our view, the usual transport equation at constant flux determines dissipation and/or fourth-order cumulants rather than third moments. Third-moments are determined in a direct fashion by (4.2), a closure equation based on the time-integrated fluctuation part of the Navier–Stokes equation.

(*f*) In a practical application context, the fluxes calculated here are not entirely the same as those previously derived by the EDQN method since coefficients differ by factors of two or three and mean shear is included. However, our coefficients are approximate (as are the previous coefficients) since they depend on an assumed spectral shape – a shape limited to large Reynolds number.

(*h*) The basic assumption of the theory is neglect of

$$\tau_0^{-1}\int_0^t \mathrm{d}t_1\,\boldsymbol{Q}^{(4)}(t-t_1)$$

which is justified if $|B|\ll 1$ and $\boldsymbol{Q}^{(4)}(0)$ is small, and B is the only missing quantity of the theory. From this perspective, it is B that is determined by comparison of theory with laboratory experiment.

A note of caution is that the entire calculation is formally limited to mean quantities that vary but little on Eulerian integral timescales and on lengthscales of energetic eddies. This limitation is common to single-point turbulence models. There seems no simple way of avoiding this limitation, but there is hope it may not be as restrictive as required by our derivation (cf. Lumley 1978; Weinstock 1986). The closure is defective for cases of very rapidly varying fluxes, such as the point-sources cases of Deardorff (1978), where the closure leads to spurious spikes of scalar concentration. However that particular defect may be corrected in the manner suggested by Deardorff – by accounting for a diffusive behaviour of $\boldsymbol{Q}^{(4)}$ when mean flux gradients are large.

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Appendix A

To verify (2.3), we take the derivative of both sides to obtain

$$\frac{\partial \boldsymbol{u}(t)}{\partial t}=\frac{\partial G_\nu(t)}{\partial t}\boldsymbol{u}(0)=\boldsymbol{I}^*(t)-\int_0^t \mathrm{d}t_1\frac{\partial G_\nu(t-t_1)}{\partial t}\boldsymbol{I}^*(t_1), \tag{A 1}$$

where $\boldsymbol{I}^*(t_1)$ denotes the square-bracket term in the integrand on the right-hand side of (2.3). From the definition (2.4) we have

$$\frac{\partial G_\nu(t-t_1)}{\partial t}=-\nu\nabla^2 G_\nu(t-t_1), \tag{A 2}$$

which is substituted in (A 1) to yield

$$\frac{\partial \boldsymbol{u}(t)}{\partial t} = -\nu\nabla^2 G_\nu(t)\,\boldsymbol{u}(0) - \boldsymbol{I}^*(t) + \nu\nabla^2 \int_0^t \mathrm{d}t_1\, G_\nu(t-t_1)\,\boldsymbol{I}^*(t_1). \tag{A 3}$$

But (2.3) is

$$\int_0^t \mathrm{d}t_1\, G_\nu(t-t_1)\,\boldsymbol{I}^*(t_1) = G_\nu(t)\,\boldsymbol{u}(0) - \boldsymbol{u}(t),$$

which we substitute in (A 3) to obtain

$$\frac{\partial \boldsymbol{u}(t)}{\partial t} = -\nu\nabla^2 \boldsymbol{u}(t) - \boldsymbol{I}^*(t), \tag{A 4}$$

which is the Navier–Stokes equation (2.1). Hence, (2.3) is equivalent to (2.1) – a formal solution of (2.1).

Appendix B

B.1. Derivation of (2.7) *and* (2.8)

To derive (2.7) and (2.8) we need Fourier expand the velocities contained in $\boldsymbol{A}(t-t_1)$. To account for turbulence inhomogeneity in this expansion it is convenient to take $\langle \boldsymbol{uu} \rangle$ proportional to $\exp(2\boldsymbol{a}\cdot\boldsymbol{x})$, where $\boldsymbol{a}$ is constant and directed along x_3 (i.e. $\boldsymbol{a}\cdot\boldsymbol{x} \equiv ax_3$). The derivation is valid for any form of weakly inhomogeneous turbulence when a is small compared to k_0, the integral wavenumber. A simple model for this $\boldsymbol{u}$ is

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}^0(\boldsymbol{x}) \exp(ax_3) \tag{B 1}$$

where $\boldsymbol{u}^0$ is statistically homogeneous. The Fourier expansion of this $\boldsymbol{u}$ is $\boldsymbol{u} = (2\Pi)^{-3} \int \mathrm{d}\boldsymbol{k} \boldsymbol{u}_k \exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x} + ax_3)$, where $\boldsymbol{k}$ is real and $\boldsymbol{u}_k$ denotes the $\boldsymbol{k}$ – Fourier transform of $\boldsymbol{u}^0(\boldsymbol{x})$.

Now, substitution of this Fourier expansion for the velocities in $\boldsymbol{A}$ we have after a little algebra

$$\int_0^t \mathrm{d}t_1\, \boldsymbol{A}(t-t_1) = \int_0^t \mathrm{d}t_1 \int \frac{\mathrm{d}\boldsymbol{k}_a\, \mathrm{d}\boldsymbol{k}_b\, \mathrm{d}\boldsymbol{k}_c\, \mathrm{d}\boldsymbol{k}_d}{(2\Pi)^{12}} \left\{(1+\mathrm{T_r})\hat{\boldsymbol{x}}_3\cdot\boldsymbol{F}(t,t_1) + \boldsymbol{F}(t,t_1)\cdot\hat{\boldsymbol{x}}_3\right\} \times \exp[4ax_3 - \nu k^2(t-t_1)], \tag{B 2}$$

$$\boldsymbol{F}(t,t_1) \equiv \langle \boldsymbol{u}_a^*(t)\boldsymbol{u}_b^*(t)[\boldsymbol{u}_c(t_1)\cdot(\mathrm{i}\boldsymbol{k}_d + \boldsymbol{a})\,\boldsymbol{u}_d(t_1)]' \rangle,$$

$$\boldsymbol{k} \equiv \boldsymbol{k}_c + \boldsymbol{k}_d,$$

where $\boldsymbol{u}_a(t)$ is the $\boldsymbol{k}_a$ Fourier transform of $\boldsymbol{u}^0(\boldsymbol{x},t)$, the asterisk denotes the complex conjugate, and $\exp[-\nu k^2(t-t_1)]$ comes from the operator $G_\nu(t-t_1)$ acting on $\exp[\mathrm{i}\boldsymbol{k}_c\cdot\boldsymbol{x} + \mathrm{i}\boldsymbol{k}_d\cdot\boldsymbol{x}]$ – with $\boldsymbol{k}_c + \boldsymbol{k}_d$ denoted by $\boldsymbol{k}$.

The fourth moment $\boldsymbol{F}$ can be expressed, approximately, in terms of second moments by the cumulant expansion

$$\langle \boldsymbol{u}_a^*\boldsymbol{u}_b^*[\boldsymbol{u}_c\cdot(\mathrm{i}\boldsymbol{k}_d + \boldsymbol{a})\boldsymbol{u}_d]' \rangle = \langle \overset{*}{\boldsymbol{u}}_a(t)\boldsymbol{u}_c(t_1) \rangle \cdot (\mathrm{i}\boldsymbol{k}_d + \boldsymbol{a}) \langle \overset{*}{\boldsymbol{u}}_b(t)\boldsymbol{u}_d(t_1) \rangle + \langle \boldsymbol{u}_a^*(t)\boldsymbol{u}_d(t_1) \rangle (\mathrm{i}\boldsymbol{k}_d + \boldsymbol{a}) \cdot \langle \boldsymbol{u}_c(t_1)\boldsymbol{u}_b^*(t) \rangle + \boldsymbol{Q}^{(4)}(t,t_1), \tag{B 3}$$

where $\boldsymbol{Q}^{(4)}$ is the fourth-order cumulant of the correlation on the left-hand side of (B 3). Substitution of (B 3) in (B 2) and using the homogeneity condition

$\langle \boldsymbol{u}_a^*(t)\boldsymbol{u}_c(t_1)\rangle = \langle \boldsymbol{u}_a^*(t)\boldsymbol{u}_a(t_1)\rangle V^{-1}\delta(\boldsymbol{k}_a-\boldsymbol{k}_c)$, (since $\boldsymbol{u}^0$ is a homogeneous field), where δ is the Dirac delta function and V^{-1} is the volume of the system, we have

$$\int_0^t \mathrm{d}t_1 \boldsymbol{A}(t,t_1) = \int_0^t \mathrm{d}t_1 \int \frac{\mathrm{d}\boldsymbol{k}_a\,\mathrm{d}\boldsymbol{k}_b}{(2\Pi)^6}(1+\mathrm{T_r})\,[\hat{\boldsymbol{x}}_3\cdot\boldsymbol{S}_a\cdot(\mathrm{i}\boldsymbol{k}_b+\boldsymbol{a})\boldsymbol{S}_b$$

$$+2\hat{\boldsymbol{x}}_3\cdot\boldsymbol{S}_a(\mathrm{i}\boldsymbol{k}_a+\boldsymbol{a})\cdot\boldsymbol{S}_b]\exp[4ax_3-\nu k^2(t-t_1)]+\int_0^t \mathrm{d}t_1\,\boldsymbol{Q}^{(4)}, \quad \text{(B 4)}$$

$$\boldsymbol{S}_a \equiv \langle \boldsymbol{u}_a^*(t)\boldsymbol{u}_a(t_1)\rangle V^{-1},$$

where $\boldsymbol{Q}^{(4)} \equiv \boldsymbol{Q}^{(4)}(t,t_1)$ denotes all the fourth-cumulant terms in the integrand of (B 2) when $\boldsymbol{F}$ is expanded as in (B 3). No attempt is made to calculate $\boldsymbol{Q}^{(4)}$ here. It is important for us to emphasize that (B 4) is a more accurate expression than is normally found in most applications of two-point closures since the main contribution to the integrals in (B 4) come from the energetic part of the spectrum at short times where two-point closures are most accurate. This is the advantage of applying two-point closures directly to evaluate single-point moments such as $\boldsymbol{A}$.

To complete the derivation of $\boldsymbol{A}$, we need the two-time moments $\boldsymbol{S}_a \equiv \boldsymbol{S}_a(t,t_1)$ for small values of $(t-t_1)$ – the large $(t-t_1)$ values do not matter much since the main contribution to the t_1 integral comes from $(t-t_1)$ less than the exponential decay time (autocorrelation time). In the absence of stratification, the small $(t-t_1)$ behaviour is given by $\boldsymbol{S}_k = \boldsymbol{S}_k^0 \exp[-\frac{1}{2}k^2v_0^2(t-t_1)^2]$ (e.g. Kraichnan 1966, 1959; Riley & Patterson 1974; Weinstock 1981) where $\boldsymbol{S}_k^0 \equiv \langle \boldsymbol{u}_k^*(t_1)\boldsymbol{u}_k(t_1)\rangle V^{-1}$ is the single-time moment and $v_0^2 \equiv \langle \boldsymbol{u}\cdot\boldsymbol{u}\rangle/3$ is mean-square fluctuating velocity along some direction. However in a stably stratified fluid the two-time moment is more complicated.

It may be approximated by (Weinstock 1986, 1978)

$$\left.\begin{aligned} &\boldsymbol{S}_k = \boldsymbol{S}_k^0 \exp[-\tfrac{1}{2}k^2v_0^2+\omega_k(t-t_1)H], \quad \text{small } |t-t_1| \\ &\omega_k^2 \equiv N^2(1-k_3^2/k^2), \end{aligned}\right\} \quad \text{(B 5)}$$

where $|t-t_1|$ is less than the decay time, ω_k is the frequency of a gravity wave whose wavevector is $\boldsymbol{k}$, k_3 denotes the vertical component of $\boldsymbol{k}$, and $H \equiv H(N^2)$ is the Heavyside step-function which is inserted to ensure that the gravity wave frequency is to be disregarded (set equal to zero) when the stratification is unstable. This expression can be verified in the limits of both weak stratification and strong stratification; i.e. in the weak stratification limit we have $k^2v_0^2 \gg \omega_k^2$ and the right-hand side of (B 5) approaches $\boldsymbol{S}_k^0 \exp[-\frac{1}{2}k^2v_0^2]$ as it should, while in the strong stratification limit we have $k^2v_0^2 \ll \omega_k^2$ and the right-hand side of (B 5) approaches $\boldsymbol{S}_k^0 \exp[\mathrm{i}\omega_k(t-t_1)]$ – a gravity wave – as it should. In the case of unstable stratification, gravity wave oscillations do not occur and $\boldsymbol{S}_k$ is the same as for the unstratified cases.

The t_1 integral in (B 4) can be evaluated approximately by substitution of (B 5) for $\boldsymbol{S}_a$ and $\boldsymbol{S}_b$ and making use of the fact the integrand has a peak maximum when $\boldsymbol{k}_a$ and $\boldsymbol{k}_b$ are in the vicinity of the characteristic wavevector $\boldsymbol{k}_0$. Afterwards, the $\boldsymbol{k}_a$ and $\boldsymbol{k}_b$ integrations can be performed by making use of

$$\int \frac{\mathrm{d}\boldsymbol{k}_a}{(2\Pi)^3}\boldsymbol{S}_a^0 \exp(2ax_3) = \langle \boldsymbol{uu}\rangle, \quad \int \frac{\mathrm{d}\boldsymbol{k}_b}{(2\Pi)^3}(\mathrm{i}\boldsymbol{k}_b+\boldsymbol{a})\boldsymbol{S}_b^0 \exp(2ax_3) = \tfrac{1}{2}\nabla\langle \boldsymbol{uu}\rangle.$$

These integrations reduce (B 4) to

$$\left.\begin{aligned}\int_0^t \mathrm{d}t_1 \boldsymbol{A}(t-t_1) &= \frac{\tau_0}{2}(1+\mathrm{T_r})(\langle u_3\boldsymbol{u}\rangle\cdot\boldsymbol{\nabla}\langle\boldsymbol{uu}\rangle + 2\langle\boldsymbol{uu}\rangle\cdot\boldsymbol{\nabla}\langle u_3\boldsymbol{u}\rangle) + \int_0^t \mathrm{d}t_1\, \boldsymbol{Q}^{(4)}(t-t_1),\\ \tau_0 &= \frac{(\Pi^{\frac{1}{2}}/4)k_0 v_0(1+R_\nu^{-1})}{\Pi N^2 H/2 + k_0^2 v_0^2(1+R_\nu^{-1})^2}, \quad R_\nu \equiv (4/\Pi)^{\frac{1}{2}} v_0/\nu k_0.\end{aligned}\right\} \tag{B 6}$$

(In arriving at (B 6) we have made a simplifying approximation for the values of $\omega_{\boldsymbol{k}_a}^2$ and $\omega_{\boldsymbol{k}_b}^2$ in the energetic part of the spectrum – an approximation for the integrated value of $(1-k_{3a}^2/k_a^2)$ and $(1-k_{3b}^2/k_b^2)$ – since these quantities depend on the anisotropy of the spectrum. These ω^2 values lie between $\frac{2}{3}N^2$, for an isotropic spectrum, to near zero for a strongly anisotropic spectrum. We have been able to justify a mean value $\frac{1}{2}N^2$ as never introducing more than a small error in τ_0 since $k_0^2 v_0^2$ is larger than $\omega_{k_0}^2$ (although $k_0 v_0$ may be less than N) for cases of interest; i.e. if $k_0^2 v_0^2$ is appreciably less than $\omega_{k_0}^2$ the turbulence is 'collapsed' and $\boldsymbol{A}^0$ and the vertical flux $\langle u_3\boldsymbol{uu}\rangle$ are practically zero.) The value of k_0 is related to ϵ, the energy dissipation rate, by means of the convenient spectral model (e.g. Reynolds 1976)

$$E(k) = \begin{cases} \alpha\epsilon^{\frac{2}{3}}k^{-\frac{5}{3}}, & k_0 \leqslant k \leqslant k_\nu \\ \alpha\epsilon^{\frac{2}{3}}k_0^{-\frac{5}{3}-m}k^m, & k \leqslant k_0, \end{cases}$$

where $\alpha \approx 1.5$ is the Kolmogoroff constant, the number m is an adjustable parameter larger than -1 that characterizes the energy-containing region of the spectrum, and k_ν is the viscous 'cut-off' wavenumber. Since $q^2 = 2\int_0^{k_\nu}\mathrm{d}kE(k)$, we have k_0 given by

$$k_0 v_0 = 6\epsilon/q^2 \tag{B 7}$$

for $m \approx 1$ and $k_\nu \gg k_0$ (large Reynolds number).

Finally, since $\langle\boldsymbol{uu}\rangle$ does not vary with x_1, or x_2, (B 6) reduces to

$$\int_0^t \mathrm{d}t_1 \boldsymbol{A}(t-t_1) = \tau_0\left[\langle u_3^2\rangle\frac{\partial}{\partial x_3}\langle\boldsymbol{uu}\rangle + (1+\mathrm{T_r})\langle u_3\boldsymbol{u}\rangle\frac{\partial}{\partial x_3}\langle u_3\boldsymbol{u}\rangle\right] + \int_0^t \mathrm{d}t_1\, \boldsymbol{Q}^{(4)}(t-t_1), \tag{B 8}$$

which is (2.7) as we set out to prove.

B.2. The derivation of other two-time correlations

The derivation of

$$\int_0^t \mathrm{d}t_i \langle u_3(t)\boldsymbol{u}(t)G_\nu(t-t_1)\boldsymbol{\nabla}p(t_1)\rangle = \tau_0\langle u_3\boldsymbol{u}\boldsymbol{\nabla}p\rangle \tag{B 9}$$

and other two-time correlations in (2.5) is similar to the derivation of (2.7) just given in §B.1. To first order in the mean field gradients, we can outline this derivation briefly as follows: we note from (2.11) that $\boldsymbol{\nabla}p$ can be divided into a 'slow' part $p^{(\mathrm{s})}$ which is second order in the velocity fluctuation, and 'fast' parts $p^{(U)}$ and $p^{(\theta)}$ which are first order in velocity fluctuation or in temperature fluctuation. The slow part gives us the correlation $\langle u_3(t)\boldsymbol{u}(t)G_\nu(t-t_1)\boldsymbol{\nabla}p^{(\mathrm{s})}(t_1)\rangle$ in (B 8) which is a fourth-moment in velocity.

Such a moment was calculated in §B.1 and the same arguments applied to our slow-term correlation gives us (B 8) with p replaced by $p^{(\mathrm{s})}$ and $\boldsymbol{Q}^{(4)}$ neglected. In other words, §B.1 gives (B 8) for the slow part of p.

For the fast velocity part $p^{(U)}$ we have $\langle u_3(t)\boldsymbol{u}(t)G_\nu(t-t_1)\nabla p^{(U)}(t_1)\rangle$ which is a third moment velocity correlation multiplied by the mean velocity gradient $\nabla \boldsymbol{U}$. Hence, to first order in mean field gradients, the third moment need only be evaluated to zeroth order. To that order, (2.3) simplifies to

$$\boldsymbol{u}(t) = \boldsymbol{u}(0) - \int_0^t \mathrm{d}t_1 [(\boldsymbol{u}\cdot\nabla\boldsymbol{u})' + \nabla p^{(\mathrm{s})}/\rho_0]$$

which upon substitution into the third moment yields a fourth moment plus an initial value term. But the initial value term is negligible when t is large, and consequently, only the fourth moment remains – to lowest order in mean field gradients when t is large. But, again, the same kind of fourth moment was calculated in §B.1 and helps us establish (B 8) for the fast part $p^{(U)}$.

The same arguments hold for the fast part $p^{(\theta)}$ since, to zeroth order in mean field gradients, the thermodynamic equation (F2) allows $\langle u_3 \boldsymbol{u} G \nabla p^{(\theta)}\rangle$ to be expressed as a fourth moment in velocity and temperature; i.e. the thermodynamic equation yields $\theta(t) = -\int \mathrm{d}t_2\,(\boldsymbol{u}\cdot\nabla\theta)$ so that $p^{(\theta)} \propto \boldsymbol{u}\cdot\nabla\theta$. Such a fourth moment is calculated in Appendix F, §F.1, and shown to be of the form (B 8) with p replaced by $p^{(\theta)}$. Hence (B 8) is valid for all parts of p to first order in mean field gradients.

In sum, we have argued that our two-time correlations can be expressed as a sum of third and fourth moments and that, to lowest order in mean field gradients, the third moments can be expressed in terms of fourth moments.

Consequently, our two-time correlations can all be expressed as fourth moments – to first order in mean field gradients – and such moments are shown to be of the form (B 8) by Appendices B and F, §§B.1 and F.1, with $\boldsymbol{Q}^{(4)}$ neglected.

Appendix C

Equations (2.15), (2.26), (2.27) and (2.28) constitute a set of linear algebraic equations for the fluxes $\langle u_3^2\rangle$, $\langle u_3^2 u_1\rangle$, $\langle u_3^2\theta\rangle$, and $\langle u_3\theta^2\rangle$ in terms of the second moments $\langle u_3^2\rangle$, $\langle u_1^2\rangle$, $\langle u_1 u_3\rangle$, $\langle u_3\theta\rangle$, $\langle u_1\theta\rangle$, and $\langle\theta^2\rangle$. The solution is obtained in a straightforward but lengthy manner by combining the equations so as to eliminate all but one flux. This procedure is repeated for each flux. For simplicity, terms of order $(g/\Theta_0)^2$ and some terms of order $(g/\Theta_0)(\partial U_0/\partial x_3)^2$ are neglected in A_4 to A_7 – the coefficients of the $\langle u_3 u_1^2\rangle$ flux. The resulting solution is given in a (convenient) matrix form by (2.29) with the coefficients A_i defined as follows:

$$A_1 = \left[1 - \frac{\tau_0^2 N^2}{3}\left(1 - \frac{4\tau_0^2 N^2}{R_1}\right)\right]\langle u_3^2\rangle - \frac{\tau_0 g}{R_1\theta_0}\left(1 - \frac{4\tau_0^2 N^2}{R_1}\right)\langle u_3\theta\rangle,$$

$$A_2 = -\frac{2\tau_0^2}{R_2}\frac{g}{\Theta_0}\left(1 - \frac{4\tau_0^2 N^2}{R_1}\right)\langle u_3^2\rangle + \frac{4\tau_0^2}{R_1 R_2}\left(\frac{g}{\Theta_0}\right)^2\langle u_3\theta\rangle,$$

$$A_3 = \frac{2\tau_0^2}{R_1 R_2}\left(\frac{g}{\Theta_0}\right)^2\langle u_3^2\rangle - \frac{4\tau_0^2}{R_1 R_2}\left(\frac{g}{\Theta_0}\right)^3\langle u_3\theta\rangle,$$

$$A_4 = \frac{2\tau_0^2}{3R_4}\left(\frac{\partial U_0}{\partial x_3}\right)^2\left(1 - \frac{9\tau_0^2 N^2}{2R_1}\right)\langle u_3^2\rangle - \frac{2\tau_0}{3R_4}\frac{\partial U_0}{\partial x_3}\left(1 - \frac{2\tau_0^2 N^2}{3R_3}\right)\left(\langle u_1 u_3\rangle + \frac{2\tau_0}{5}\frac{\partial U_0}{\partial x_3}\langle u_3^2\rangle\right),$$

$$A_5 = \frac{1}{R_4}\langle u_3^2\rangle - \frac{\tau_0}{3R_4}\frac{g}{\Theta_0}\langle u_3\theta\rangle,$$

$$A_6 = \frac{2}{R_4}\langle u_1 u_3\rangle \frac{2\tau_0}{R_4}\frac{\partial U_0}{\partial x_3}\left(1-\frac{2\tau_0^2 N^2}{3R_3}\right)\langle u_3^2\rangle + \frac{2\tau_0^2}{3R_4}\frac{g}{\Theta_0}\frac{\partial U_0}{\partial x_3}\langle u_3\theta\rangle,$$

$$A_7 = -\frac{2\tau_0}{3R_4}\frac{g}{\Theta_0}\langle u_1 u_3\rangle,$$

$$A_8 = \frac{1}{R_2}\left(1-\frac{4\tau_0^2 N^2}{R_1}\right)\langle \theta u_3\rangle - \frac{\tau_0}{R_2}\frac{\partial \theta_0}{\partial x_3}\left(1-\frac{4\tau_0^2 N^2}{R_1}\right)\langle u_3^2\rangle,$$

$$A_9 = \frac{\langle u_3^2\rangle}{R_2} - \frac{4\tau_0}{R_1 R_2}\frac{g}{\Theta_0}\langle u_3\theta\rangle,$$

$$A_{10} = -\frac{2\tau_0}{R_1 R_2}\frac{g}{\Theta_0}\left(\langle u_3^2\rangle - \tau_0\frac{g}{\Theta_0}\langle u_3\theta\rangle\right),$$

$$A_{11} = \frac{2\tau_0}{R_1}\frac{\partial \Theta_0}{\partial x_3}\left[1-\frac{4\tau_0^2 N^2}{R_2}\left(1-\frac{4\tau_0^2 N^2}{R_1}\right)\right]\left(\tau_0\frac{\partial \Theta_0}{\partial x_3}\langle u_3\rangle - \langle u_3\theta\rangle\right),$$

$$A_{12} = \frac{2}{R_1}\left(1+\frac{4\tau_0^4 N^4}{R_1 R_2}\right)\langle \theta u_3\rangle - \frac{4\tau_0}{R_1}\frac{\partial \Theta_0}{\partial x_3}\left[1-\frac{\tau_0^2 N^2}{R_2}\left(1-\frac{4\tau_0^2 N^2}{R_1}\right)\right]\langle u_3^2\rangle,$$

$$A_{13} = \frac{1}{R_1}\left(1+\frac{4\tau_0^4 N^4}{R_1 R_2}\right)\left(\langle u_3^2\rangle - \tau_0\frac{g}{\Theta_0}\langle u_3\theta\rangle\right),$$

$$R_1 = 3+\tfrac{7}{3}\tau_0^2 N^2, \quad R_2 = 3+\tau_0^2 N^2\left(1-\frac{4\tau_0^2 N^2}{R_1}\right), \quad R_3 = 3+\tfrac{4}{3}\tau_0^2 N^2,$$

$$R_4 = 3+\tfrac{1}{3}\tau_0^2 N^2 - \frac{2\tau_0^3}{9}N^2\frac{\partial U_0}{\partial x_3}.$$

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